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A graph is $K(1,r)-free if it does not contain K(1,r) as an induced subgraph. It is claw-free if it does not contain K(1,3) as an induced subgraph. Matthews and Sumner [5] proved that every 2-connected, claw-free graph with min. degree at least (p-2)/3 is Hamiltonian. In this paper we investigate Hamilton cycles in K(1,r)-free graphs with respect to a minimum degree condition.

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HAMILTONIAN RESULTS IN $K_{1,r}$-FREE GRAPHS

by

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ABSTRACT

A graph is $K_{1,r}$*-free* if it does not contain $K_{1,r}$ as an induced subgraph. It is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. Matthews and Sumner [5] proved that every 2-connected, claw-free graph with $\delta \geq (p-2)/3$ is Hamiltonian. In this paper we investigate Hamilton cycles in $K_{1,r}$-free graph with respect to a minimum degree condition.

Preliminaries

A graph is $K_{1,r}$-free if it does not contain $K_{1,r}$ as an induced subgraph. A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. There are many sufficient conditions for a graph to be Hamiltonian. One of the oldest is due to Dirac [3] which gives a sufficient condition in terms of the minimum degree δ.

Theorem 1[3]
Let G be a graph with $p \geq 3$ and

$$\delta \geq p/2.$$

Then G is Hamiltonian. □

Ore [7] generalised this result by looking at the degree sum of 2 nonadjacent vertices.

Theorem 2[7]
Let G be a graph with $p \geq 3$ and

$$\deg u + \deg v \geq p$$

for all nonadjacent pairs of vertices u, v. Then G is Hamiltonian. □

Bondy [1] looked at the degree sums of triples of mutually nonadjacent vertices.

Theorem 3[1]
Let G be a 2-connected graph with

$$\deg u + \deg v + \deg w \geq 3p/2$$

for all independent triples of vertices, u, v, w. Then G is Hamiltonian. □

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Results

In this paper, we look at minimum degree conditions for $K_{1,r}$-free graphs to be Hamiltonian. The first result, due to Matthews and Sumner [5] is for claw-free graphs.

Theorem 4[5]
Let G be a 2-connected, claw-free graph with

$$\delta \geq (p-2)/3.$$

Then G is Hamiltonian. □

This result is sharp, as shown by the family of graphs in Figure 1 which are 2-connected, claw-free with $\delta = (p-3)/3$, but are not Hamiltonian.

The next two results are both due to Nash-Williams [6].

Theorem 5[6]
Let G be a 2-connected graph with

$$\delta \geq (p+2)/3$$

and $p \geq 3$, and let C be a longest cycle of G. Then no two vertices of $G-C$ are adjacent. □

This bound on δ is sharp, for example the non-Hamiltonian graph $K_2 + 3K_n$, for $n \geq 2$ has $\delta = (p+1)/3$ and $G-C$ has at least one edge for any longest cycle C.

Lemma 6[6]
Let G be a 2-connected graph with

$$\delta \geq (p+2)/3.$$

Then if $\alpha \leq \delta$, G is Hamiltonian.

This bound on α is sharp as shown by the non-Hamiltonian graphs $K_{n,n-1}$, $n \geq 4$. These graphs have $\alpha = n = \delta + 1$, with $\delta = n-1 \geq (p+2)/3$. The bound on δ is also sharp, as shown by the non-Hamiltonian graphs $K_2 + 3K_n$, $n \geq 2$. These graphs have $\delta = (p+1)/3$ and $\alpha = 3 \leq \delta$.

A classical result, due to Chvátal and Erdős [2] that relates α not to δ but to the connectivity, κ, is the following.

Theorem 7[2]
Let G be a connected graph with

$$\alpha \leq \kappa.$$

Then G is Hamiltonian. □

Lemma 6 does not follow from Theorem 7. For example, the graph $2K_1 + 2K_n$ with the edges of a K_m ($m \leq (p+2)/6$) removed from each K_n has $\kappa = 2$ and $\alpha = 2m$ and $\delta \geq (p+2)/3$. Thus α can be much larger than κ with the graph still being Hamiltonian.

The idea of a bound on α will be employed in the proof of Theorem 8, which is a similar result to Theorem 4, but using $K_{1,4}$-free graphs instead of $K_{1,3}$-free.

Theorem 8

Let G be a 2-connected $K_{1,4}$-free graph with

$$\delta \geq (p+2)/3.$$

Then G is Hamiltonian.

Proof

We will first show that $\alpha \leq \delta$ and use Lemma 6.

Suppose, to the contrary, that $\alpha \geq \delta + 1$. Let T denote any largest independent set in G and let $\alpha = |T|$. The number of edges from T to $G - T$ is at least $\delta\alpha$. The number of edges from $G - T$ to T is at most $3|V(G) - T| = 3(p - \alpha)$ since G is $K_{1,4}$-free. We get the inequality

$$\delta\alpha \leq 3(p - \alpha)$$

and so

$$\alpha \leq 3p/(\delta + 3).$$

Now $\alpha \geq \delta + 1$ and so

$$\begin{aligned} \delta + 1 &\leq 3p/(\delta + 3) \\ (\delta + 1)(\delta + 3) &\leq 3p \\ (p+5)(p+11)/9 &\leq 3p \\ p^2 + 16p + 55 &\leq 27p \\ p^2 - 11p + 55 &\leq 0. \end{aligned}$$

But $p^2 - 11p + 55$ is positive for all real p. Thus $\alpha \leq \delta$ and by Lemma 6, G is Hamiltonian. □

The bound on the minimum degree cannot be lowered. For any n, the non-Hamiltonian graphs $K_2 + 3K_n$ are $K_{1,4}$-free, have $\delta = (n-1) + 2 = n + 1 = (p+1)/3$.

Next, we will look at the case for $K_{1,r}$-free graphs. First we need the following theorem. A bipartite graph is said to be *balanced* if it has bipartition $X \cup Y$ and $|X| = |Y|$. The following result, due to Jackson [4] will be used in the proof of Theorem 10.

Theorem 9[4]

Let G be a balanced bipartite graph with

$$\delta \geq (p+2)/4.$$

Then G is Hamiltonian. □

We will now prove the analogue of Theorem 8 for $K_{1,r}$-free graphs with $r \geq 5$.

Theorem 10

Let G be a 2-connected, $K_{1,r}$-free graph, $r \geq 5$, with

$$\delta \geq (p + r - 3)/3.$$

Then G is Hamiltonian unless $p = 2r - 3$. If $p = 2r - 3$, then G is Hamiltonian unless $G - E(G - T)$ is $K_{r-1,r-2}$ where T is any largest independent set of G.

Proof

Let G be a $K_{1,r}$-free graph with $\delta \geq (p + r - 3)/3$, and suppose that G is not Hamiltonian. Clearly $\delta \geq (p + 2)/3$ since $r \geq 5$. Let T denote any largest independent set in G, so that $|T| = \alpha$. Then since G is not Hamiltonian we have $\alpha \geq \delta + 1$ by Lemma 6. The number of edges from T to $G - T$ is at least $\delta\alpha$ and the number of edges from $G - T$ to T is at most $(r - 1)(p - \alpha)$, since G is $K_{1,r}$-free. We get

$$\begin{aligned} \delta\alpha &\leq (r-1)(p-\alpha) \\ \alpha &\leq (r-1)p/(\delta + r - 1). \end{aligned}$$

Now $\alpha \geq \delta + 1$, so we get

$$\begin{aligned} \delta + 1 &\leq (r-1)p/(\delta + r - 1) \\ (\delta + 1)(\delta + r - 1) &\leq (r-1)p. \end{aligned}$$

So by hypothesis,

$$\begin{aligned} (p + 4r - 6)(p + r) &\leq 9(r-1)p \\ p^2 + (3 - 4r)p + 4r^2 - 6r &\leq 0 \\ (p - 2r)(p - 2r + 3) &\leq 0. \end{aligned}$$

Thus if G is not Hamiltonian, we must have $2r - 3 \leq p \leq 2r$.

Suppose $p = 2r$. Then $\delta \geq (2r + r - 3)/3 = r - 1$. Now as above, T is any largest independent set in G and the number of edges from T to $G - T$ is at least $\delta\alpha \geq (r-1)\alpha$. The number of edges from $G - T$ to T is at most $(r - 1)(p - \alpha) = (r - 1)(2r - \alpha)$. So we get

$$(r-1)\alpha \leq (r-1)(2r - \alpha)$$

and so

$$\alpha \leq r.$$

Thus if G is not Hamiltonian we have $\delta = r - 1$ and $\alpha = r$, and the number of edges from T to $G - T$ is at least $\alpha\delta = r(r - 1)$. The number of edges from $G - T$ to T is at most $(2r - \alpha)(r - 1) = r(r - 1)$ so there are precisely $r(r - 1)$ edges between T and $G - T$. So each vertex in T is adjacent to $r - 1$ vertices of $G - T$ and each vertex of $G - T$ is adjacent to $r - 1$ vertices of T.

Consider the graph $H = G - E(G - T)$. This is a balanced bipartite graph with $\delta = r - 1 \geq (p+2)/4$. So by Theorem 9, H is Hamiltonian and therefore so is G.

Next suppose $p = 2r - 1$. Then $\delta \geq (2r - 1 + r - 3)/3$ and so $\delta \geq r - 1$. Let T be any largest independent set, and so the number of edges from T to $G-T$ is at least $\delta\alpha \geq (r-1)\alpha$. The number of edges from $G - T$ to T is at most $(r-1)(p-\alpha) = (r-1)(2r-1-\alpha)$ since G is $K_{1,r}$-free. We get

$$(r-1)\alpha \leq (r-1)(2r-1-\alpha)$$
$$\alpha \leq (2r-1)/2$$

and since α is an integer

$$\alpha \leq r - 1.$$

Thus $\alpha \leq \delta$ and by Lemma 6, G is Hamiltonian.

Now suppose $p = 2r - 2$. Then $\delta \geq (2r - 2 + r - 3)/3$ and so $\delta \geq r - 1$. Now let T be any largest independent set. The number of edges from T to $G-T$ is at least $\delta\alpha \geq (r-1)\alpha$. The number of edges from $G - T$ to T is at most $(r-1)(p-\alpha) = (r-1)(2r-2-\alpha)$. We get

$$(r-1)\alpha \leq (r-1)(2r-2-\alpha).$$

So

$$\alpha \leq (2r-2)/2 = r - 1.$$

Thus $\alpha \leq \delta$ and G is Hamiltonian.

Finally suppose $p = 2r - 3$. Then $\delta \geq (2r - 3 + r - 3)/3 = r - 2$. Let T be any largest independent set. The number of edges from T to $G - T$ is at least $\delta\alpha \geq (r-2)\alpha$. The number of edges from $G - T$ to T is at most $(r-1)(p-\alpha) = (r-1)(2r-3-\alpha)$. We get

$$(r-2)\alpha \leq (r-1)(2r-3-\alpha)$$
$$(2r-3)\alpha \leq (r-1)(2r-3)$$
$$\alpha \leq r - 1.$$

So if G is not Hamiltonian we must have $\alpha = r - 1$ and $\delta = r - 2$. The number of edges from T to $G-T$ is at least $(r-1)(r-2)$ so that each vertex of T is adjacent to every vertex of $G - T$ and each vertex of $G - T$ is adjacent to every vertex of T. Then $G - E(G - T)$ is the non-Hamiltonian bipartite graph $K_{r-1,r-2}$, and hence G is not Hamiltonian. □

The bound on δ in Theorem 10 cannot be reduced. This is shown by the $K_{1,r}$-free graph $K_{r-2,r-3}$. This graph is non-Hamiltonian and has $\delta = r - 3 = (p + r - 4)/3$. Also, by adding edges in this graph to the smaller of the two bipartition sets we get additional non-Hamiltonian graphs with $\delta = (p + r - 4)/3$.

As an example of the exceptional graphs mentioned in this theorem, take the $K_{1,5}$-free case. Here, there are precisely 4 exceptional graphs all with $p = 7$. These are obtained from $K_{3,4}$ by adding 0,1,2 or 3 edges to the smaller partition. (See Figure 2).

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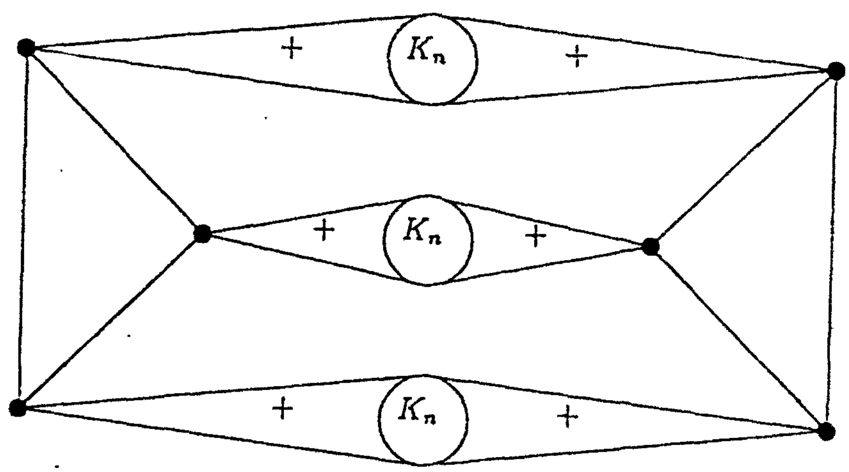

Figure 1

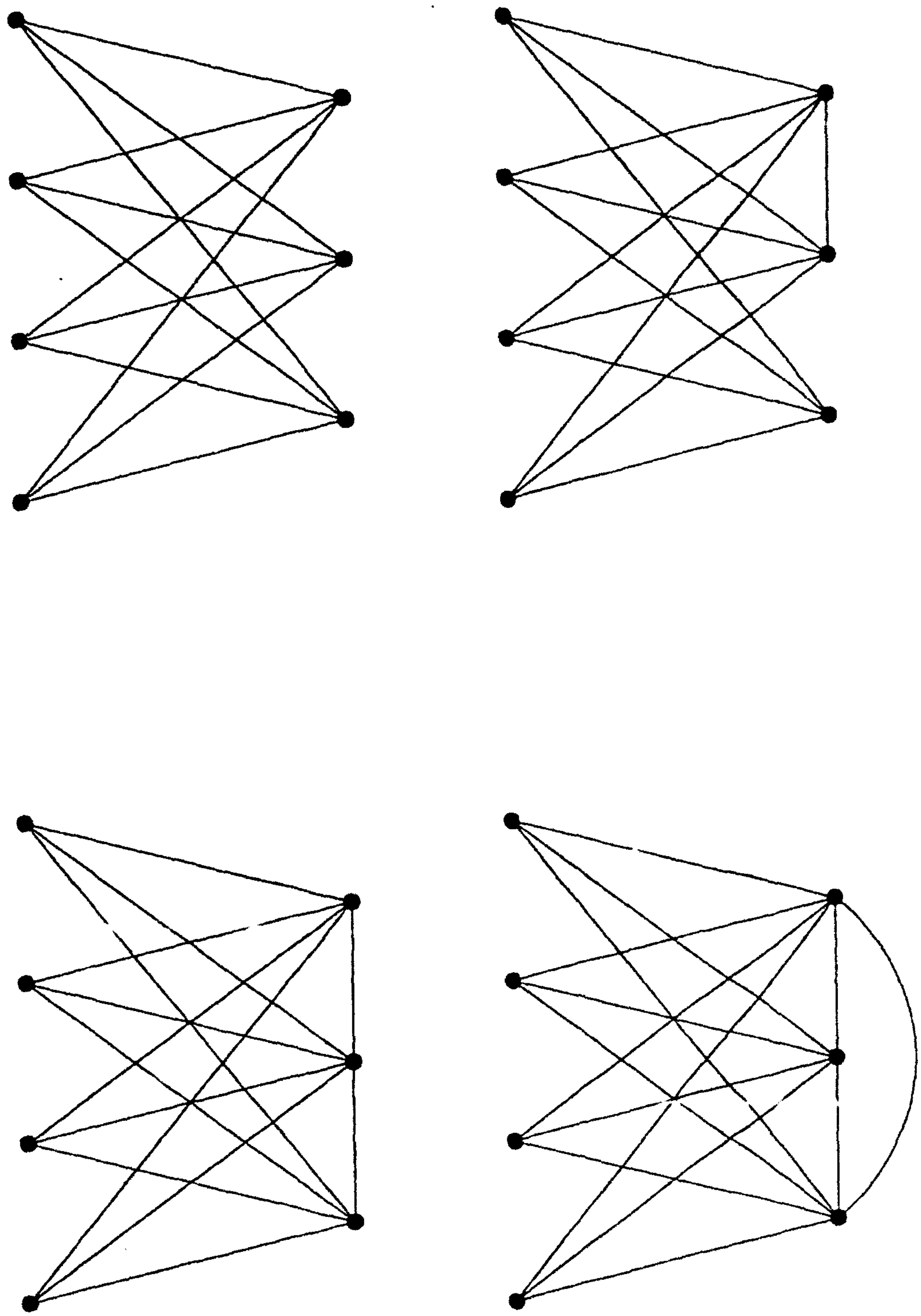

Figure 2